Recitation 10: Characteristic Functions and Weak Convergence

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Exercise 1. *Prove that random variable X is symmetric (X and $-X$ have the same law) if and only if its characteristic function φ_X takes real value.*

Exercise 2. *Calculate the characteristic function for the random variable X if*

1. *X follows Bernoulli distribution of parameter $p \in (0,1)$;*
2. *X follows Binominal distribution of parameter (n,p);*
3. *X follows Poisson distribution of parameter λ;*
4. *X follows exponential distribution of parameter θ;*
5. *X follows symmetric exponential distribution of density $f(y) = \frac{\lambda}{2}e^{-\lambda|y|}$;*
6. *X follows Cauchy distribution of density $f(x) = \frac{\alpha}{\pi(\alpha^2+x^2)}$.*

Exercise 3. *Let $X \sim \mathcal{N}(0,\sigma^2)$ and Φ its characteristic function.*

1. *Prove that $\Phi'(t) = -t\sigma^2\Phi(t)$;*
2. *Calculate $\Phi(t)$.*

Exercise 4. *Prove that the sum of independent Gaussian (Poisson, Cauchy) random variables are Gaussian (Poisson, Cauchy).*

Exercise 5 (Total variation convergence)**.** *We define the total variation distance between two random discrete random variables X and Y that*

$$d_{TV}(X,Y) = \sup_{A \in \mathbb{Z}} |\mathbb{P}[X \in A] - \mathbb{P}[Y \in A]|.$$

1. *Prove an equivalent definition that*

$$d_{TV}(X,Y) = \frac{1}{2}\sum_{z\in\mathbb{Z}} |\mathbb{P}[X = z] - \mathbb{P}[Y = z]|.$$

2. *Prove that if $d_{TV}(X_n, X) \xrightarrow{n\to\infty} 0$, then $X_n \xrightarrow{d} X$.*
3. *Prove that for X_1, X_2 independent, Y_1, Y_2 independent, then*

$$d_{TV}(X_1 + X_2, Y_1 + Y_2) \leqslant d_{TV}(X_1, Y_1) + d_{TV}(X_2, Y_2).$$

4. *Use total variation distance to prove that, for independent Bernoulli random variables $X_{n,i}$ of parameter $p_{n,i}$, if*
   - $\sum_i p_{n,i} \xrightarrow{n\to\infty} \lambda$;
   - $\max_i p_{n,i} \xrightarrow{n\to\infty} 0$;

   then we have $\sum_i X_{n,i} \xrightarrow[n\to\infty]{d}$ Poisson(λ).